On the satisfiability problem for a 3-level quantified syllogistic

Domenico Cantone and Marianna Nicolosi Asmundo

Università di Catania, Dipartimento di Matematica e Informatica
email: cantone@dmi.unict.it, nicolosi@dmi.unict.it

Abstract. We show that a collection of three-sorted set-theoretic formulae, denoted $3LQS^R$ and which admits a restricted form of quantification over individual and set variables, has a solvable satisfiability problem by proving that it enjoys a small model property, i.e., any satisfiable $3LQS^R$-formula ψ has a finite model whose size depends solely on the size of ψ itself. We also introduce the sublanguages $(3LQS^R)^h$ of $3LQS^R$, whose formulae are characterized by having quantifier prefixes of length bounded by $h \geq 2$ and some other syntactic constraints, and we prove that each of them has the satisfiability problem NP-complete. Then, we show that the modal logic S5 can be formalized in $(3LQS^R)^3$.

1 Introduction

Computable set theory is a research field active since the late seventies. Its initial goal was the design of effective decision procedures to be implemented in theorem provers/verifiers, for larger and larger collections of set-theoretic formulae (also called *syllogistics*). During the years, however, due to the production of several decidability results of a purely theoretical nature the main emphasis shifted to the foundational goal of narrowing the boundary between the decidable and the undecidable in set theory.

The main results in computable set theory up to 2001 have been collected in [9,10]. We also mention that the most efficient decision procedures have been implemented in the proof verifier ÆtnaNova [15,16] and within one of versions of the system STeP [2].

The basic set-theoretic fragment is the so-called Multi-Level Syllogistic (*MLS*, for short) which involves in addition to variables varying over the von Neumann universe of sets and to propositional connectives also the basic set-theoretic operators such as $\cup$, $\cap$, $\setminus$, and the predicates $=$, $\in$, and $\subseteq$. MLS was proved decidable in [13] and extended over the years in several ways by the introduction of various operators, predicates, and restricted forms of quantification.

Most of the decidability results in computable set theory deal with one-sorted multi-level syllogistics, namely collections of formulae involving variables of one type only, ranging over the von Neumann universe of sets. On the other hand, few decidability results have been found for multi-sorted stratified syllogistics, where variables of several types are allowed. This, despite of the fact that in

many fields of computer science and mathematics often one deals with multi-sorted languages.

An efficient decision procedure for the satisfiability of the Two-Level Syllogistic language ($2LS$), a version of MLS with variables of two sorts for individuals and sets of individuals, has been presented in [12]. Subsequently, in [5], the extension of $2LS$ with the singleton operator and the Cartesian product operator has been proved decidable. The result has been obtained by embedding $2LS$ in the class of purely universal formulae of the elementary theory of relations. Tarski's and Presburger's arithmetics extended with sets have been studied in [7]. The three-sorted language $3LSSPU$ (Three-Level Syllogistic with Singleton, with Powerset and general Union) has been proved decidable in [6]. More specifically, $3LSSPU$ has three types of variables, ranging over individuals, sets of individuals, and collections of sets of individuals, respectively, and involves the singleton, powerset, and general union operators, in addition to the operators and predicates present in $2LS$.

In this paper we present a decidability result for the satisfiability problem of the set-theoretic language $3LQS^R$ (Three-Level Quantified Syllogistic with Restricted quantifiers), which is a three-sorted quantified syllogistic involving *individual variables*, varying over the elements of a given nonempty universe D, *set variables*, ranging over subsets of D, and *collection variables*, varying over collections of subsets of D.

The language of $3LQS^R$ admits a restricted form of quantification over individual and set variables. Its vocabulary contains only the predicate symbols $=$ and $\in$. In spite of that, $3LQS^R$ allows to express several constructs of set theory. Among them, the most comprehensive one is the set former, which in turn allows to express other operators like the powerset operator, the singleton operator, and so on.

We will prove that $3LQS^R$ enjoys a small model property by showing how one can extract, out of a given model satisfying a $3LQS^R$-formula ψ, another model of ψ of bounded finite cardinality. The construction of the finite model is inspired to the algorithms described in [12], [5], and [6].

Then, we introduce the sublanguages $(3LQS^R)^h$,of $3LQS^R$, consisting of $3LQS^R$-formulae having the quantifier prefixes of size bounded by $h \geq 2$ and satisfying some further syntactic constraints. It is shown that each $(3LQS^R)^h$ has the satisfiability problem NP-complete and that $(3LQS^R)^3$ can express the normal modal logic $\mathsf{S5}$.

The paper is organized as follows. In Section 2, we introduce the language $3LQS^R$ and we illustrate its expressiveness. Subsequently, in Section 3 the machinery needed to prove the decidability result is provided. In particular, a general definition of a relativized $3LQS^R$-interpretation is introduced, together with some useful technical results. In Section 4, the small model property for $3LQS^R$ is established, thus solving the satisfiability problem for $3LQS^R$. Then, in Section 5, after some examples illustrating the expressivity of $3LQS^R$ in set theory, we introduce the sublanguages $(3LQS^R)^h$, show that they have a NP-complete sat-

isfiability problem, and that $(3LQS^R)^3$ can express the modal logic S5. Finally, in Section 6, we draw our conclusions.

2 The language $3LQS^R$

We present the language $3LQS^R$ of our interest as follows. We begin by defining in Section 2.1 the syntax and the semantics of a more general three-level quantified language, denoted $3LQS$, which contains $3LQS^R$ as a proper fragment. Then, in Section 2.2, we characterize $3LQS^R$ by means of suitable restrictions on the usage of quantifiers in formulae of $3LQS$.

2.1 The more general language $3LQS$

Syntax of $3LQS$ The three-level quantified language $3LQS$ involves[1]

(i) a collection $\mathcal{V}_0$ of *individual* or *sort 0* variables, denoted by lower case letters $x, y, z, \ldots$;
(ii) a collection $\mathcal{V}_1$ of *set* or *sort 1* variables, denoted by final upper case letters $X, Y, Z, \ldots$;
(iii) a collection $\mathcal{V}_2$ of *collection* or *sort 2* variables, denoted by initial upper case letters $A, B, C, \ldots$.

The atomic formulae of $3LQS$ are defined as follows:

(1) level 0 atomic formulae:
- $x = y$, for $x, y \in \mathcal{V}_0$;
- $x \in X$, for $x \in \mathcal{V}_0, X \in \mathcal{V}_1$;

(2) level 1 atomic formulae:
- $X = Y$, for $X, Y \in \mathcal{V}_1$;
- $X \in A$, for $X \in \mathcal{V}_1, A \in \mathcal{V}_2$;
- $(\forall z_1) \ldots (\forall z_n)\varphi_0$, with φ_0 a propositional combination of level 0 atoms and $z_1, \ldots, z_n$ variables of sort 0;

(3) level 2 atomic formulae:
- $(\forall Z_1) \ldots (\forall Z_m)\varphi_1$, where φ_1 is a propositional combination of level 0 and level 1 atoms, and $Z_1, \ldots, Z_m$ are variables of sort 1.

Finally, the formulae of $3LQS$ are all the propositional combinations of atoms of level $0, 1$, and 2.

[1] In the paper, variables often come with numerical subscripts. Other types of subscripts are used in Section 5 for variables denoting sets or collections of sets of particular relevance (i.e., X_U, $A_{\pi,h}$).

Semantics of 3*LQS* A 3*LQS-interpretation* is a pair $\boldsymbol{\mathcal{M}} = (D, M)$, where

– D is any nonempty collection of objects, called the *domain* or *universe* of $\boldsymbol{\mathcal{M}}$, and
– M is an assignment over variables of $3LQS$ such that
  • $Mx \in D$, for each individual variable $x \in \mathcal{V}_0$;
  • $MX \in \text{pow}(D)$, for each set variable $X \in \mathcal{V}_1$;
  • $MA \in \text{pow}(\text{pow}(D))$, for all collection variables $A \in \mathcal{V}_2$.[2]

Let

- $\boldsymbol{\mathcal{M}} = (D, M)$ be a $3LQS$-interpretation,
- $x_1, \ldots, x_l \in \mathcal{V}_0$,
- $X_1, \ldots, X_m \in \mathcal{V}_1$,
- $u_1, \ldots, u_l \in D$,
- $U_1, \ldots, U_m \in \text{pow}(D)$,

By

$$\boldsymbol{\mathcal{M}}[x_1/u_1, \ldots, x_l/u_l, X_1/U_1, \ldots, X_m/U_m,]\,,$$

we denote the interpretation $\boldsymbol{\mathcal{M}}' = (D, M')$ such that

$$\begin{aligned} M'x_i &= u_i\,, \text{ for } i = 1, \ldots, l \\ M'X_j &= U_j\,, \text{ for } j = 1, \ldots, m \end{aligned}$$

and which otherwise coincides with $\boldsymbol{\mathcal{M}}$ on all remaining variables. Throughout the paper we use the abbreviations: $\mathcal{M}^z$ for $\boldsymbol{\mathcal{M}}[z_1/u_1, \ldots, z_n/u_n]$ and $\mathcal{M}^Z$ for $\boldsymbol{\mathcal{M}}[Z_1/U_1, \ldots, Z_m/U_m]$.

Definition 1. *Let φ be a $3LQS$-formula and let $\boldsymbol{\mathcal{M}} = (D, M)$ be a $3LQS$-interpretation. We define the notion of* satisfiability *of φ with respect to $\boldsymbol{\mathcal{M}}$ (denoted by $\boldsymbol{\mathcal{M}} \models \varphi$) inductively as follows*

1. $\boldsymbol{\mathcal{M}} \models x = y$ *iff* $Mx = My$;
2. $\boldsymbol{\mathcal{M}} \models x \in X$ *iff* $Mx \in MX$;
3. $\boldsymbol{\mathcal{M}} \models X = Y$ *iff* $MX = MY$;
4. $\boldsymbol{\mathcal{M}} \models X \in A$ *iff* $MX \in MA$;
5. $\boldsymbol{\mathcal{M}} \models (\forall z_1) \ldots (\forall z_n)\varphi_0$ *iff* $\boldsymbol{\mathcal{M}}[z_1/u_1, \ldots, z_n/u_n] \models \varphi_0$, *for all* $u_1, \ldots, u_n \in D$;
6. $\boldsymbol{\mathcal{M}} \models (\forall Z_1) \ldots (\forall Z_m)\varphi_1$ *iff* $\boldsymbol{\mathcal{M}}[Z_1/U_1, \ldots, Z_m/U_m] \models \varphi_1$, *for all* $U_1, \ldots, U_n \in \text{pow}(D)$.
   Propositional connectives are interpreted in the standard way, namely
7. $\boldsymbol{\mathcal{M}} \models \varphi_1 \wedge \varphi_2$ *iff* $\boldsymbol{\mathcal{M}} \models \varphi_1$ *and* $\boldsymbol{\mathcal{M}} \models \varphi_2$;
8. $\boldsymbol{\mathcal{M}} \models \varphi_1 \vee \varphi_2$ *iff* $\boldsymbol{\mathcal{M}} \models \varphi_1$ *or* $\boldsymbol{\mathcal{M}} \models \varphi_2$;
9. $\boldsymbol{\mathcal{M}} \models \neg\varphi$ *iff* $\boldsymbol{\mathcal{M}} \not\models \varphi$. □

Let ψ be a $3LQS$-formula, if $\boldsymbol{\mathcal{M}} \models \psi$, i.e. $\boldsymbol{\mathcal{M}}$ *satisfies* ψ, then $\boldsymbol{\mathcal{M}}$ is said to be a $3LQS$-*model* for ψ. A $3LQS$-formula is said to be *satisfiable* if it has a $3LQS$-model. A $3LQS$-formula is *valid* if it is satisfied by all $3LQS$-interpretations.

[2] We recall that, for any set s, $\text{pow}(s)$ denotes the *powerset* of s, i.e., the collection of all subsets of s.

2.2 Characterizing $3LQS^R$

$3LQS^R$ is the subcollection of the formulae ψ of $3LQS$ such that, for *every* atomic formula $(\forall Z_1), \ldots, (\forall Z_m)\varphi_1$ of level 2 occurring in ψ and *every* level 1 atomic formula of the form $(\forall z_1) \ldots (\forall z_n)\varphi_0$ occurring in φ_1, the condition

$$\neg\varphi_0 \rightarrow \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} z_i \in Z_j \tag{1}$$

is a valid $3LQS$-formula (in this case we say that the atom $(\forall z_1) \ldots (\forall z_n)\varphi_0$ is *linked* to the variables $Z_1, \ldots, Z_m$).

Condition (1) guarantees that, if a given interpretation assigns to $z_1, \ldots, z_n$ elements of the domain that make φ_0 false, such values are contained in at least one of the subsets of the domain assigned to $Z_1, \ldots, Z_m$. As shown in the proof of statement (ii) of Lemma 4, this fact is used to make sure that satisfiability is preserved in the finite model. As the examples in Section 5 illustrate, condition (1) is not particularly restrictive.

The following question arises: how one can establish whether a given $3LQS$-formula is a $3LQS^R$-formula? Observe that condition (1) involves no collection variables and no quantification. Indeed, it turns out to be a $2LS$-formula and therefore one could use the decision procedures in [12] to test its validity, since $3LQS$ is a conservative extension of $2LS$. We mention also that in most cases of interest, as will be shown in detail in Section 5, condition (1) is just an instance of the simple propositional tautology $\neg(A \rightarrow B) \rightarrow A$, and therefore its validity can be established just by inspection.

Finally, we observe that though the semantics of $3LQS^R$ plainly coincides with the one given above for $3LQS$-formulae, nevertheless we will refer to $3LQS$-interpretations of $3LQS^R$-formulae as $3LQS^R$-*interpretations*.

3 Relativized interpretations

We introduce the notion of relativized interpretation, to be used together with the decision procedure of Section 4.2 to construct, out of a model $\boldsymbol{\mathcal{M}} = (D, M)$ for a $3LQS^R$-formula ψ, a finite interpretation $\boldsymbol{\mathcal{M}}^* = (D^*, M^*)$ of bounded size satisfying ψ as well.

Definition 2. *Let $\boldsymbol{\mathcal{M}} = (D, M)$ be a $3LQS^R$-interpretation. Let $D^* \subseteq D$, $d^* \in D^*$, and $\mathcal{V}'_1 \subseteq \mathcal{V}_1$. The* relativized *interpretation* $\mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}'_1)$ *of $\boldsymbol{\mathcal{M}}$ with respect to D^*, d^*, and $\mathcal{V}'_1$ is the interpretation (D^*, M^*) such that*

$$M^*x = \begin{cases} Mx\,, & \textit{if } Mx \in D^* \\ d^*\,, & \textit{otherwise} \end{cases}$$

$$M^*X = MX \cap D^*$$

$$M^*A = ((MA \cap \mathrm{pow}(D^*)) \setminus \{M^*X : X \in \mathcal{V}'_1\}) \cup \{M^*X : X \in \mathcal{V}'_1,\ MX \in MA\}\,.$$

□

The definition of relativized interpretation given above is inspired by the construction of the finite model described in [12], [5], and in [6]. We spend some words on the intuition behind the definition of M^*A. Analogously to M^*X, M^*A is obtained from the intersection of the interpretation of A in $\boldsymbol{\mathcal{M}}$ with the power set of the finite domain D^*. However, such operation may leave in $MA \cap \mathrm{pow}(D^*)$ some sets J such that $J = M^*X$ but $MX \notin MA$. Such J's have to be removed from the restricted interpretation of A in order to guarantee that satisfiability of ψ is preserved. Further, there also may be some $MX \in MA$ such that $M^*X \notin MA \cap \mathrm{pow}(D^*)$. Again, to let the restricted model preserve satisfiability of ψ, such M^*X have to be added to the interpretation of A in the restricted model.

For ease of notation, we will often omit the reference to the element $d^* \in D^*$ and write simply $\mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, \mathcal{V}'_1)$ in place of $\mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}'_1)$.

The following satisfiability result holds for unquantified atomic formulae.

Lemma 1. *Let $\boldsymbol{\mathcal{M}} = (D, M)$ be a $3LQS^R$-interpretation. Also, let $D^* \subseteq D$, $d^* \in D^*$, and $\mathcal{V}'_1 \subseteq \mathcal{V}_1$ be given. Let us put $\boldsymbol{\mathcal{M}}^* = \mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, d^*, \mathcal{V}'_1)$. Then the following holds.*

(a) $\boldsymbol{\mathcal{M}}^* \models x = y$ iff $\boldsymbol{\mathcal{M}} \models x = y$, for all $x, y \in \mathcal{V}_0$ such that $Mx, My \in D^*$;
(b) $\boldsymbol{\mathcal{M}}^* \models x \in X$ iff $\boldsymbol{\mathcal{M}} \models x \in X$, for all $X \in \mathcal{V}_1$ and $x \in \mathcal{V}_0$ such that $Mx \in D^*$;
(c) $\boldsymbol{\mathcal{M}}^* \models X = Y$ iff $\boldsymbol{\mathcal{M}} \models X = Y$, for all $X, Y \in \mathcal{V}_1$ such that if $MX \neq MY$ then $(MX \Delta MY) \cap D^* \neq \emptyset$;
(d) if for all $X, Y \in \mathcal{V}'_1$ such that $MX \neq MY$ we have $(MX \Delta MY) \cap D^* \neq \emptyset$, then $\boldsymbol{\mathcal{M}}^* \models X \in A$ iff $\boldsymbol{\mathcal{M}} \models X \in A$, for all $X \in \mathcal{V}'_1$, $A \in \mathcal{V}_2$.[3]

Proof.

Cases (a), (b) and (c) are easily verified. We prove only case (d). To this end, assume that for all $X, Y \in \mathcal{V}'_1$ such that $MX \neq MY$ we have $(MX \Delta MY) \cap D^* \neq \emptyset$. Let $X \in \mathcal{V}'_1$ and $A \in \mathcal{V}_2$. If $MX \in MA$, then obviously $M^*X \in M^*A$. On the other hand, if $MX \notin MA$, but $M^*X \in M^*A$, then we must necessarily have $M^*X = M^*Z$, for some $Z \in \mathcal{V}'_1$ such that $MZ \in MA$. But then, as $MX \neq MZ$, from our hypothesis we would obtain $M^*X \neq M^*Z$, which is a contradiction.

■

3.1 Relativized interpretations and quantified atomic formulae

Satisfiability results for quantified atomic formulae are treated as shown in the following. Let us put

$$\begin{aligned}\boldsymbol{\mathcal{M}}^{z,*} &= \mathrm{Rel}(\boldsymbol{\mathcal{M}}^z, D^*, \mathcal{V}'_1)\\ \boldsymbol{\mathcal{M}}^{*,z} &= \boldsymbol{\mathcal{M}}^*[z_1/u_1, \ldots, z_n/u_n]\end{aligned}$$

[3] We recall that Δ denotes the symmetric difference operator defined by $s \Delta t = (s \setminus t) \cup (t \setminus s)$.

$$\mathcal{M}^{Z,*} = \text{Rel}(\mathcal{M}^{Z}, D^*, \mathcal{V}_1' \cup \{Z_1, \ldots, Z_m\})$$
$$\mathcal{M}^{*,Z} = \mathcal{M}^*[Z_1/U_1, \ldots, Z_m/U_m].$$

The following lemmas provide useful technical results to be employed in the proof of Theorem 1 below. In particular, Lemmas 2 and 3, which are simply stated without proof, are used to prove Lemma 4.

Lemma 2. *Let $u_1, \ldots, u_n \in D^*$ and let $z_1, \ldots, z_n \in \mathcal{V}_0$. Then, for every $x \in \mathcal{V}_0$ and $X \in \mathcal{V}_1$ we have:*

(i) $M^{*,z}x = M^{z,*}x$,
(ii) $M^{z,*}X = M^{*,z}X$.

Lemma 3. *Let $\mathcal{M} = (D, M)$ be a $3LQS^R$-interpretation, $D^* \subseteq D$, $\mathcal{V}_1' \subseteq \mathcal{V}_1$, $Z_1, \ldots, Z_m \in \mathcal{V}_1 \setminus \mathcal{V}_1'$, $U_1, \ldots, U_m \in \text{pow}(D^*) \setminus \{M^*X : X \in \mathcal{V}_1'\}$.*
Then the $3LQS^R$-interpretations $\mathcal{M}^{*,Z}$ and $\mathcal{M}^{Z,*}$ coincide.

Lemma 4. *Let $\mathcal{M} = (D, M)$ be a $3LQS^R$-interpretation. Let $D^* \subseteq D$, $d^* \in D^*$, $\mathcal{V}_1' \subseteq \mathcal{V}_1$ be given, let $\mathcal{M}^* = \text{Rel}(\mathcal{M}, D^*, d^*, \mathcal{V}_1')$. Further, let $(\forall z_1) \ldots (\forall z_n)\varphi_0$ and $(\forall Z_1) \ldots (\forall Z_m)\varphi_1$ be atomic formulae of level 1 and 2, respectively, such that $Mx \in D^*$, for every $x \in \mathcal{V}_0$ occurring in φ_0 or in φ_1. Then we have*

(i) if $\mathcal{M} \models (\forall z_1) \ldots (\forall z_n)\varphi_0$, then $\mathcal{M}^* \models (\forall z_1) \ldots (\forall z_n)\varphi_0$;
(ii) if $\mathcal{M} \models (\forall Z_1) \ldots (\forall Z_m)\varphi_1$, then $\mathcal{M}^* \models (\forall Z_1) \ldots (\forall Z_m)\varphi_1$, provided that
- *$(MX \Delta MY) \cap D^* \neq \emptyset$, for every $X, Y \in \mathcal{V}_1$ with $MX \neq MY$, and that*
- *there are $u_1, \ldots, u_n \in D^*$ such that $\mathcal{M}[v_1/u_1, \ldots, v_n/u_n] \not\models \varphi_0$, for every $(\forall z_1) \ldots (\forall z_n)\varphi_0$ not satisfied by $\mathcal{M}$, and occurring in ${\varphi_1}^{Z_1,\ldots,Z_m}_{X_1,\ldots,X_m}$, with $X_1, \ldots, X_m$ variables in $\mathcal{V}_1'$.*

Proof.

(i) Assume by contradiction that there exist $u_1, \ldots, u_n \in D^*$ such that $\mathcal{M}^{*,z} \not\models \varphi_0$. Then, there must be an atomic formula φ_0' in φ_0 (either of type $x = y$ or $x \in X$) that is differently interpreted in $\mathcal{M}^{*,z}$ and in $\mathcal{M}^z$.
Let us suppose first that φ_0' is the atom $x = y$ and, without loss of generality, that $\mathcal{M}^{*,z} \not\models x = y$. By Lemma 2, we have $M^{z,*}x \neq M^{z,*}y$. Since $M^{z,*}x = M^z x$, $M^{z,*}y = M^z y$, and, by hypothesis, $M^z x = M^z y$, we obtain a contradiction.
Now let us suppose that φ_0 is the atom $x \in X$ and, without loss of generality, assume that $\mathcal{M}^{*,z} \not\models x \in X$. By Lemma 2, we have $M^{z,*}x \notin M^{z,*}X$, that is $M^z x \notin M^z X \cap D^*$, again a contradiction.

(ii) Assume, by way of contradiction, that $\mathcal{M}^* \not\models (\forall Z_1) \ldots (\forall Z_m)\varphi_1$. Hence there exist $U_1, \ldots, U_m \in pow(D^*)$ such that $\mathcal{M}^{*,Z} \not\models \varphi_1$.
Without loss of generality, assume that $U_i = M^*X_i$, for $1 \leq i \leq k$ ($k \geq 0$) for some variables $X_1, \ldots, X_k$ in $\mathcal{V}_1'$, and that $U_j \neq M^*X$ for all $k+1 \leq j \leq m$ and for all variables X in $\mathcal{V}_1'$.

Let $\bar{\varphi}_1$ be the formula obtained by simultaneously substituting $Z_1, \ldots, Z_k$ with $X_1, \ldots, X_k$ in φ_1, and let $\boldsymbol{\mathcal{M}}^{*,Z_k} = \boldsymbol{\mathcal{M}}^*[Z_{k+1}/U_{k+1}, \ldots, Z_m/U_m]$. Further, let $\mathcal{M}^{Z'}$ be a $3LQS^R$-interpretation differing from $\boldsymbol{\mathcal{M}}^Z$ only in the evaluation of $Z_1, \ldots, Z_k$ ($M^{Z'}Z_1 = MX_1, \ldots, M^{Z'}Z_k = MX_k$).
Now we can distinguish two cases.

If $k = m$, then $\boldsymbol{\mathcal{M}}^{*,Z_k}$ and $\boldsymbol{\mathcal{M}}^*$ coincide and a contradiction can be obtained by showing that the implications

$$\boldsymbol{\mathcal{M}}^{*,Z} \not\models \varphi_1 \Rightarrow \boldsymbol{\mathcal{M}}^* \not\models \bar{\varphi}_1 \Rightarrow \boldsymbol{\mathcal{M}} \not\models \bar{\varphi}_1 \Rightarrow \boldsymbol{\mathcal{M}}^{Z'} \not\models \varphi_1$$

hold. Hence, against the hypothesis, we get that $\boldsymbol{\mathcal{M}} \not\models (\forall Z_1) \ldots (\forall Z_m)\varphi_1$.
The first implication, $\boldsymbol{\mathcal{M}}^{*,Z} \not\models \varphi_1 \Rightarrow \boldsymbol{\mathcal{M}}^* \not\models \bar{\varphi}_1$, is plainly derived from the definition of $\bar{\varphi}_1$. The second one, $\boldsymbol{\mathcal{M}}^* \not\models \bar{\varphi}_1 \Rightarrow \boldsymbol{\mathcal{M}} \not\models \bar{\varphi}_1$, can be proved by showing that $\boldsymbol{\mathcal{M}}^*$ and $\boldsymbol{\mathcal{M}}$ interpret each atomic formula $\bar{\varphi}'_1$ occurring in $\bar{\varphi}_1$ in the same manner.
If $\bar{\varphi}'_1$ is an atomic formula of level 0 or an atomic formula of level 1 of type $X = Y$ and $X \in A$, the proof follows directly from Lemma 1.
If $\bar{\varphi}'_1$ is an atomic formula of level 1 of type $(\forall z_1) \ldots, (\forall z_n)\varphi_0$, the implication $\boldsymbol{\mathcal{M}} \models (\forall z_1) \ldots (\forall z_n)\varphi_0 \Rightarrow \boldsymbol{\mathcal{M}}^* \models (\forall z_1) \ldots (\forall z_n)\varphi_0$ follows from statement (i) of the lemma, whereas $\boldsymbol{\mathcal{M}}^* \models (\forall z_1) \ldots (\forall z_n)\varphi_0 \Rightarrow \boldsymbol{\mathcal{M}} \models (\forall z_1) \ldots (\forall z_n)\varphi_0$ can be proved by contradiction as follows. Assume that $\boldsymbol{\mathcal{M}} \not\models (\forall z_1) \ldots (\forall z_n)\varphi_0$. Then, by hypothesis, there are $u_1, \ldots, u_n$ in D^* such that $\boldsymbol{\mathcal{M}}[z_1/u_1, \ldots, z_n/u_n] \not\models \varphi_0$ and, by Lemmas 1, 2, $\boldsymbol{\mathcal{M}}^*[z_1/u_1, \ldots, z_n/u_n] \not\models \varphi_0$ contradicting our hypothesis.
The last implication, $\boldsymbol{\mathcal{M}} \not\models \bar{\varphi}_1 \Rightarrow \boldsymbol{\mathcal{M}}^{Z'} \not\models \varphi_1$, is deduced by the definition of $\bar{\varphi}_1$ and of Z'.

If $k < m$, the schema of the proof is analogous to the previous case. However, since $\boldsymbol{\mathcal{M}}^{*,Z_k}$ and $\boldsymbol{\mathcal{M}}^*$ do not coincide, the single steps are carried out in a slightly different manner. Thus, for the sake of clarity we report the proof below.
In order to obtain a contradiction we prove that

$$\boldsymbol{\mathcal{M}}^{*,Z} \not\models \varphi_1 \Rightarrow \boldsymbol{\mathcal{M}}^{*,Z_k} \not\models \bar{\varphi}_1 \Rightarrow \boldsymbol{\mathcal{M}}^{Z_k} \not\models \bar{\varphi}_1 \Rightarrow \boldsymbol{\mathcal{M}}^{Z'} \not\models \varphi_1$$

hold.
The first implication $\boldsymbol{\mathcal{M}}^{*,Z} \not\models \varphi_1 \Rightarrow \boldsymbol{\mathcal{M}}^{*,Z_k} \not\models \bar{\varphi}_1$ can be immediately deduced from the definition of $\bar{\varphi}_1$ and of $\boldsymbol{\mathcal{M}}^{*,Z_k}$. The second implication $\boldsymbol{\mathcal{M}}^{*,Z_k} \not\models \bar{\varphi}_1 \Rightarrow \boldsymbol{\mathcal{M}}^{Z_k} \not\models \bar{\varphi}_1$ can be proved by showing that every atomic formula $\bar{\varphi}'_1$ in $\bar{\varphi}_1$ is interpreted in $\boldsymbol{\mathcal{M}}^{*,Z_k}$ and in $\boldsymbol{\mathcal{M}}^{Z_k}$ in the same way.
The proof is straightforwardly carried out using Lemmas 3 and 1 in case $\bar{\varphi}'_1$ is an atomic formula of level 0, or an atomic formula of level 1 of type $X = Y$ and $X \in A$.
If $\bar{\varphi}'_1$ is an atomic formula of level 1 of type $(\forall z_1) \ldots (\forall z_n)\varphi_0$, we first show that $\boldsymbol{\mathcal{M}}^{Z_k} \models (\forall z_1) \ldots (\forall z_n)\varphi_0$ implies that $\boldsymbol{\mathcal{M}}^{*,Z_k} \models (\forall z_1) \ldots (\forall z_n)\varphi_0$.
If $\boldsymbol{\mathcal{M}}^{Z_k} \models (\forall z_1) \ldots (\forall z_n)\varphi_0$, we have that $\boldsymbol{\mathcal{M}}^{Z_k,*} \models (\forall z_1) \ldots (\forall z_n)\varphi_0$, by (i) of the present lemma, and that $\boldsymbol{\mathcal{M}}^{*,Z_k} \models (\forall z_1) \ldots (\forall z_n)\varphi_0$, by Lemma 3.

Now, let us show that $\boldsymbol{\mathcal{M}}^{*,Z_k} \models (\forall z_1)\dots(\forall z_n)\varphi_0$ implies that $\boldsymbol{\mathcal{M}}^{Z_k} \models (\forall z_1)\dots(\forall z_n)\varphi_0$.

Assume by contradiction that $\boldsymbol{\mathcal{M}}^{Z_k} \not\models (\forall z_1)\dots(\forall z_n)\varphi_0$. Then there exist $u_1,\dots,u_n \in D$ such that $\boldsymbol{\mathcal{M}}^{Z_k}[z_1/u_1,\dots,z_n/u_n] \not\models \varphi_0$. In particular, by the condition $\neg\varphi_0 \to \wedge_{i=1}^{n} \vee_{j=k+1}^{m} z_i \in Z_j$, we derive that $u_1,\dots,u_n$ are elements of D^*. This allows us to apply Lemma 1 and to obtain that $\boldsymbol{\mathcal{M}}^{Z_k,z,*} \not\models \varphi_0$. Then, by Lemma 2 we obtain $\boldsymbol{\mathcal{M}}^{Z_k,*,z} \not\models \varphi_0$ and hence $\boldsymbol{\mathcal{M}}^{Z_k,*} \not\models (\forall z_1)\dots(\forall z_n)\varphi_0$. Thus, Lemma 3 yields $\boldsymbol{\mathcal{M}}^{*,Z_k} \not\models (\forall z_1)\dots(\forall z_n)\varphi_0$ contradicting the hypothesis.

Finally, the third implication, $\boldsymbol{\mathcal{M}}^{Z_k} \not\models \bar{\varphi}_1 \Rightarrow \boldsymbol{\mathcal{M}}^{Z'} \not\models \varphi_1$ is directly derived from the definition of $\bar{\varphi}_1$ and of Z'. ■

4 The satisfiability problem for $3LQS^R$-formulae

In this section we solve the satisfiability problem for $3LQS^R$, i.e. the problem of establishing for any given formula of $3LQS^R$ whether it is satisfiable or not, as follows:

(a) firstly, we reduce effectively the satisfiability problem for $3LQS^R$-formulae to the satisfiability problem for normalized $3LQS^R$-conjunctions (these will be defined precisely below);
(b) secondly, we prove that the collection of normalized $3LQS^R$-conjunctions enjoys a small model property.

From (a) and (b), the solvability of the satisfiability problem for $3LQS^R$ follows immediately. Additionally, by further elaborating on point (a), it could easily be shown that indeed the whole collection of $3LQS^R$-formulae enjoys a small model property.

4.1 Normalized $3LQS^R$-conjunctions

Let ψ be a formula of $3LQS^R$ and let ψ_{DNF} be a disjunctive normal form of ψ. Then ψ is satisfiable if and only if at least one of the disjuncts of ψ_{DNF} is satisfiable. We recall that the disjuncts of ψ_{DNF} are conjunctions of level $0,1$, and 2 literals, i.e. level $0,1$, and 2 atoms or their negation. In view of the previous observations, without loss of generality, we can suppose that our formula ψ is a conjunction of level $0,1$, and 2 literals. In addition, we can also assume that no bound variable in ψ can occur in more than one quantifier or can occur also free.

For decidability purposes, negative quantified conjuncts occurring in ψ can be eliminated as explained below. Let $\boldsymbol{\mathcal{M}} = (D, M)$ be a model for ψ. Then $\boldsymbol{\mathcal{M}} \models \neg(\forall z_1)\dots(\forall z_n)\varphi_0$ if and only if $\mathcal{M}[z_1/u_1,\dots,z_n/u_n] \models \neg\varphi_0$, for some $u_1,\dots,u_n \in D$, and $\boldsymbol{\mathcal{M}} \models \neg(\forall Z_1)\dots(\forall Z_m)\varphi_1$ if and only if $\boldsymbol{\mathcal{M}}[Z_1/U_1,\dots,Z_m/U_m] \models \neg\varphi_1$, for some $U_1,\dots,U_m \in \mathrm{pow}(D)$. Thus, each negative literal of type $\neg(\forall z_1)\dots(\forall z_n)\varphi_0$ can be replaced by $\neg(\varphi_0)^{z_1,\dots,z_n}_{z'_1,\dots,z'_n}$, where

$z'_1, \ldots, z'_n$ are newly introduced variables of sort 0. Likewise, each negative literal of type $\neg(\forall Z_1) \ldots (\forall Z_m)\varphi_1$ can be replaced by $\neg(\varphi_1)^{Z_1,\ldots,Z_m}_{Z'_1,\ldots,Z'_m}$, where $Z'_1, \ldots, Z'_m$ are newly introduced variables of sort 1.

Hence, we can further assume that ψ is a conjunction of literals of the following types:

(1) $x = y$, $\neg(x = y)$, $x \in X$, $\neg(x \in X)$, $X = Y$, $\neg(X = Y)$, $X \in A$, $\neg(X \in A)$;
(2) $(\forall z_1) \ldots (\forall z_n)\varphi_0$, where $n > 0$ and φ_0 is a propositional combination of level 0 atoms;
(3) $(\forall Z_1) \ldots (\forall Z_m)\varphi_1$, where $m > 0$ and φ_1 is a propositional combination of level 0 and level 1 atoms, where atoms of type $(\forall z_1) \ldots (\forall z_n)\varphi_0$ in φ_1 are linked to the bound variables $Z_1, \ldots, Z_m$.

We call such formulae *normalized* $3LQS^R$*-conjunctions*.

4.2 A small model property for normalized $3LQS^R$-conjunctions

In view of the preceding discussion we can limit ourselves to consider the satisfiability problem for normalized $3LQS^R$-conjunctions only.

Thus, let ψ be a normalized $3LQS^R$-conjunction and assume that $\boldsymbol{\mathcal{M}} = (D, M)$ is a model for ψ.

We show how to construct, out of $\boldsymbol{\mathcal{M}}$, a finite $3LQS^R$-interpretation $\boldsymbol{\mathcal{M}}^* = (D^*, M^*)$ which is a model of ψ. We proceed as follows. First we outline a procedure to build a nonempty finite universe $D^* \subseteq D$ whose size depends solely on ψ and can be computed *a priori*. Then, a finite $3LQS^R$-interpretation $\boldsymbol{\mathcal{M}}^* = (D^*, M^*)$ is constructed according to Definition 2. Finally, we show that $\boldsymbol{\mathcal{M}}^*$ satisfies ψ.

Construction of the universe D^*. Let us denote by $\mathcal{W}_0$, $\mathcal{W}_1$, and $\mathcal{W}_2$ the collections of the variables of sort 0, 1, and 2 present in ψ, respectively. Then we compute D^* by means of the procedure described below.

Let $\psi_1, \ldots, \psi_k$ be the conjuncts of ψ. To each conjunct ψ_i of the form $(\forall Z_{i1}) \ldots (\forall Z_{im_i})\varphi_i$ we associate the collection $\varphi_{i1}, \ldots, \varphi_{i\ell_i}$ of atomic formulae of type (2) present in the matrix of ψ_i and call the variables $Z_{i1}, \ldots, Z_{im_i}$ the *arguments of* $\varphi_{i1}, \ldots, \varphi_{i\ell_i}$. Then we put

$$\Phi = \{\varphi_{ij} : 1 \leq i \leq k \text{ and } 1 \leq j \leq \ell_i\}.$$

For every pair of variables X, Y in $\mathcal{W}_1$ such that $MX \neq MY$, let $u_{X,Y}$ be any element in the symmetric difference of MX and MY and put $\Delta_1 = \{u_{XY} : X, Y \text{ in } \mathcal{W}_1 \text{ and } MX \neq MY\}$. If Δ_1 is constructed applying the procedure *Distinguish* described in [8], it holds that $|\Delta_1| \leq |\mathcal{W}_1| - 1$.

We initialize D^* with the set $\{Mx : x \text{ in } \mathcal{W}_0\} \cup \Delta_1$. Then, for each $\varphi \in \Phi$ of the form $(\forall z_1) \ldots (\forall z_n)\varphi_0$ having $Z_1, \ldots, Z_m$ as arguments and for each ordered m-tuple $(X_{i_1}, \ldots, X_{i_m})$ of variables in $\mathcal{W}_1$, if $\mathcal{M}{\varphi_0}^{Z_1\ ,\ldots,\ Z_m}_{X_{i_1},\ldots,X_{i_m}} = \mathbf{false}$ we insert in D^* elements $u_1, \ldots, u_n \in D$ such that

$$\boldsymbol{\mathcal{M}}[z_1/u_1, \ldots, z_n/u_n]{\varphi_0}^{Z_1\ ,\ldots,\ Z_m}_{X_{i_1},\ldots,X_{i_m}} = \mathbf{false}\,,$$

otherwise we leave D^* unchanged.

From the previous construction it easily follows that

$$|D^*| \leq |\mathcal{W}_0| + |\mathcal{W}_1| - 1 + ((|\mathcal{W}_1|^{\max m}) \max n)|\Phi| \,, \tag{2}$$

where $\max m$ and $\max n$ are respectively the maximal number of quantifiers in formulae of level 2 and the maximal number of quantifiers in formulae of level 1 occurring in quantified formulae of level 2. Thus, in general, the domain of the small model D^* is exponential in the size of the input formula ψ.

Correctness of the relativization. Let us put $\boldsymbol{\mathcal{M}}^* = \mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, \mathcal{W}_1)$. We have to show that, if $\boldsymbol{\mathcal{M}} \models \psi$, then $\boldsymbol{\mathcal{M}}^* \models \psi$.

Theorem 1. *Let $\boldsymbol{\mathcal{M}}$ be a $3LQS^R$-interpretation satisfying a normalized $3LQS^R$-conjunction ψ. Further, let $\boldsymbol{\mathcal{M}}^* = \mathrm{Rel}(\boldsymbol{\mathcal{M}}, D^*, \mathcal{W}_1)$ be the $3LQS^R$-interpretation defined according to Definition 2, where D^* is constructed as described above, and let $\mathcal{W}_1$ be defined as above. Then $\boldsymbol{\mathcal{M}}^* \models \psi$.*

Proof.

We have to prove that $\boldsymbol{\mathcal{M}}^* \models \psi'$ for every literal ψ' in ψ. Each ψ' is of one of the three types introduced in Section 4.1. By applying Lemma 1 or 4 to every ψ' in ψ (according to the type of ψ') we obtain the thesis.

Notice that the hypotheses of Lemma 1 and of Lemma 4 are fulfilled by the construction of D^* outlined above:

- $Mx \in D^*$, for every variable $x \in \mathcal{V}_0$. Furthermore, $(MX \Delta MY) \cap D^* \neq \emptyset$ for every $X, Y \in \mathcal{V}_1$ such that $MX \neq MY$ (one just needs to substitute the generic set of individual variables $\mathcal{V}_0$ with $\mathcal{W}_0$ and $\mathcal{V}_1$ with $\mathcal{W}_1$);
- for every atomic formula of type $(\forall z_1) \ldots (\forall z_n)\varphi_0$ occurring in an atomic formula of level 2 and such that $\boldsymbol{\mathcal{M}} \not\models (\forall z_1) \ldots (\forall z_n)\varphi_0$, there are $u_1, \ldots, u_n$ elements of D^* such that $\boldsymbol{\mathcal{M}}[z_1/u_1, \ldots, z_n/u_n] \not\models \varphi_0$. ■

From the above reduction and relativization steps, it is not hard to derive the following result:

Corollary 1. *The fragment $3LQS^R$ enjoys a small model property (and therefore its satisfiability problem is solvable).* ■

5 Expressiveness of the language $3LQS^R$

Several constructs of elementary set theory are easily expressible within the language $3LQS^R$. In particular, as will be shown below, it is possible to express with $3LQS^R$-formulae a restricted variant of the set former, which in turn allows to express other significant set operators such as binary union, intersection, set difference, the singleton operator, the powerset operator and its variants, etc.

(1) $X = Y$, $X \subseteq Y$	(5) $\neg(X = Y)$, $\neg(X \subseteq Y)$
(2) $X = Y \cap Z$, $X = Y \cup Z$	(6) $x \in X$
(3) $X = \overline{Y}$	(7) $\neg(x = y)$
(4) $X = 0$, $X = 1$	(8) $x = y$

Table 1. $2LS$ literals.

More specifically, atomic formulae of type $X = \{z : \varphi(z)\}$ can be expressed in $3LQS^R$ by the formula

$$(\forall z)(z \in X \leftrightarrow \varphi(z))\,, \tag{3}$$

provided that after transforming it into prenex normal form one obtains a formula satisfying the syntactic constraints of $3LQS^R$. In particular, this is always the case whenever $\varphi(z)$ is any unquantified formula of $3LQS^R$.

The same remark applies also to atomic formulae of type $A = \{Z : \varphi(Z)\}$. In this case, in order for a prenex normal form of

$$(\forall Z)(Z \in A \leftrightarrow \varphi(Z)) \tag{4}$$

to be in the language $3LQS^R$, it is enough that

(a) $\varphi(Z)$ does not contain any quantifier over variables of sort 1, and
(b) all quantified variables of sort 0 in $\varphi(Z)$ are linked to the variable Z as specified in condition (1).

In what follows we introduce the stratified syllogistics $2LS$, already mentioned in the introduction, and $3LSSP$ (Three-Level Syllogistic with Singleton and Powerset), and describe their formalization in $3LQS^R$. Then we show how to express some other set-theoretical constructs. Finally, in Section 5.4 we introduce a family of sublanguages of $3LQS^R$ having the satisfiability problem NP-complete and able to express the modal logic S5.

5.1 Two-Level Syllogistic

$2LS$ is a fragment of the elementary theory of sets admitting individual variables, $x, y, z, \ldots$, set variables, $X, Y, Z, \ldots$, and the constants 0 and 1 standing respectively for the empty set and the domain of the discourse. Terms and formulae of $2LS$ are constructed out of variables and constants by means of the set operators of union, intersection, and set complementation, the binary relators $=$, $\in$, and $\subseteq$, and the propositional connectives.

$2LS$ has been proved decidable in [12] by a procedure that, taking as input a conjunction φ of literals of the forms illustrated in Table 1, stops with failure in case φ is unsatisfiable, otherwise returns a model for φ.

(1) $A = B$, $A \subseteq B$	(4) $A = 0$, $A = 1$
(2) $A = B \cap C$, $A = B \cup C$	(5) $\neg(A = B)$, $\neg(A \subseteq B)$
(3) $A = \overline{A}$	(6) $X \in A$

Table 2. $3LS$ literals.

(1) $X = \{x\}$	(2) $A = \{X\}$	(3) $A = \mathrm{pow}(X)$

Table 3. Literals with singleton and powerset set operators.

Every literal from Table 1 can readily be expressed as a formula of $3LQS^R$. Indeed, $X = Y$ is an atomic formula of level 1 of $3LQS^R$, whereas $X \subseteq Y$ can be expressed by the quantified atomic formula $(\forall z)(z \in X \rightarrow z \in Y)$ of level 1. $X = Y \cup Z$ can be translated into the formula $(\forall z)((z \in Y \vee z \in Z) \leftrightarrow z \in X)$ and $X = Y \cap Z$ into $(\forall z)((z \in Y \wedge z \in Z) \leftrightarrow z \in X)$. $X = \overline{Y}$ can be expressed by $(\forall z)(z \in X \leftrightarrow \neg(z \in Y))$. Literals of type $X = 0$ and $X = 1$ are translated in the atomic formulae of level 1 $(\forall z)\neg(z \in X)$ and $(\forall z)(z \in X)$, respectively. Literals of $2LS$ of type $(6), (7)$, and (8) are just atomic formulae of $3LQS^R$ of level 0.

5.2 Three Level Syllogistic with Singleton and Powerset

$3LSSP$ is the sublanguage of $3LSSPU$ not involving the set theoretic construct of general union. It can be obtained from $2LS$ by extending it with a new sort of variables $A, B, C, \ldots$, ranging over collections of sets. Furthermore, besides the usual set theoretical constructs, $3LSSP$ involves the set singleton operator $\{\cdot\}$ and the powerset operator pow.

$3LSSP$ can plainly be decided by the decision procedure presented in [6] for the whole $3LSSPU$.

All formulae in Tables 2 and 3 are readily expressible by $3LQS^R$-formulae. For instance, $A = B$ can be translated into the $3LQS^R$-formula $(\forall Z)(Z \in A \leftrightarrow Z \in B)$ of level 2, whereas $A \subseteq B$ can be formalized as $(\forall Z)(Z \in A \rightarrow Z \in B)$. The literals $A = B \cap C$ and $A = B \cup C$ can be translated into $(\forall Z)(Z \in A \leftrightarrow (Z \in B \wedge Z \in C))$ and $(\forall Z)(Z \in A \leftrightarrow (Z \in B \vee Z \in C))$, respectively. $A = \overline{B}$ can be expressed by $(\forall Z)(Z \in A \leftrightarrow \neg(Z \in B))$. Literals of type $A = 0$ and $A = 1$ can be expressed by the formulae $(\forall Z)\neg(Z \in A)$ and $(\forall Z)(Z \in A)$, respectively. Literals of type (6) are just atomic $3LQS^R$-formulae of level 1.

The singleton of level 1, $X = \{x\}$, is expressed by the atomic formula $(\forall z)(z \in X \leftrightarrow z = x)$ of level 1, whereas the singleton of level 2, $A = \{X\}$, is translated into the formula $(\forall Z)(Z \in A \leftrightarrow Z = X)$ of level 2. Finally, the powerset of a set X, $A = \mathrm{pow}(X)$, is translated into the formula $\varphi \equiv (\forall Z)(Z \in A \leftrightarrow$

$(\forall z)(z \in Z \rightarrow z \in X))$. It is easy to check that φ satisfies the restriction on quantifiers introduced in Section 2.2. In fact, putting $\varphi_0 \equiv (z \in Z \rightarrow z \in X)$ and considering that the general expression $\bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} z_i \in Z_j$ in this case just reduces to $z \in Z$, we have that the condition $\neg\varphi_0 \rightarrow \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} z_i \in Z_j$ is instantiated to $\neg(z \in Z \rightarrow z \in X) \rightarrow z \in Z$, which is an instance of the tautological schema $\neg(A \rightarrow B) \rightarrow A$.

5.3 Other set theoretical constructs expressible in $3LQS^R$

Other constructs of set theory are expressible in the $3LQS^R$ formalism.

For instance, the literal $A = \mathrm{pow}_{\leq h}(X)$, where $\mathrm{pow}_{\leq h}(X)$ denotes the collection of all the subsets of X having at most h distinct elements, can be expressed in $3LQS^R$ as

$$\begin{aligned}(\forall Z)(Z \in A \leftrightarrow ((\forall z)(z \in Z \rightarrow z \in X) \\ \wedge(\forall z_1)\ldots(\forall z_{h+1})(\textstyle\bigwedge_{i=1}^{h+1} z_i \in Z \\ \rightarrow \neg(\textstyle\bigwedge_{i=1}^{h+1} \bigwedge_{j=1, j\neq i}^{h+1} \neg(z_i = z_j))))).\end{aligned}$$

Likewise, the literals $A = \mathrm{pow}_{<h}(X)$ and $A = \mathrm{pow}_{=h}(X)$, where $\mathrm{pow}_{<h}(X)$ and $\mathrm{pow}_{=h}(X)$ denote, respectively, the collection of subsets of X with less than h elements and the collection of subsets of X with exactly h distinct elements, can be expressed in an analogous way.

In the formalization of $A = \mathrm{pow}_{\leq h}(X)$ given above, the restriction on quantifiers of Section 2.2 is satisfied. This can easily be verified for both conjuncts

$\varphi_1 \equiv (\forall z)(z \in Z \rightarrow z \in X)$, and
$\varphi_2 \equiv (\forall z_1)\ldots(\forall z_{h+1})(\bigwedge_{i=1}^{h+1} z_i \in Z \rightarrow \neg(\bigwedge_{i=1}^{h+1} \bigwedge_{j=1, j\neq i}^{h+1} \neg(z_i = z_j)))$.

The verification of the validity of condition (1) for φ_1 is identical to the one shown for the formula considered in the previous paragraph. Thus we just check its validity for φ_2.

One just needs to put $\varphi_0 \equiv \bigwedge_{i=1}^{h+1} z_i \in Z \rightarrow \neg(\bigwedge_{i=1}^{h+1} \bigwedge_{j=1, j\neq i}^{h+1} \neg(z_i = z_j))$ and observe that $\bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} z_i \in Z_j$ is just $\bigwedge_{i=1}^{h+1} z_i \in Z$. Thus $\neg\varphi_0 \rightarrow \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} z_i \in Z_j$ is just the formula

$$\neg(\bigwedge_{i=1}^{h+1} z_i \in Z \rightarrow \neg(\bigwedge_{i=1}^{h+1} \bigwedge_{j=1, j\neq i}^{h+1} \neg(z_i = z_j))) \rightarrow \bigwedge_{i=1}^{h+1} z_i \in Z$$

which again is plainly an instance of the propositional tautology $\neg(A \rightarrow B) \rightarrow A$.

The Cartesian product $A = X_1 \otimes \ldots \otimes X_n$ can be formalized by the $3LQS^R$-formula

$$\begin{aligned}(\forall Z)(Z \in A \leftrightarrow ((\forall z_1)\ldots(\forall z_n)(\textstyle\bigwedge_{i=1}^{n} z_i \in Z \rightarrow \bigwedge_{i=1}^{n} z_i \in X_i) \\ \wedge(\forall z_1)\ldots(\forall z_{n+1})(\textstyle\bigwedge_{i=1}^{n+1} z_i \in Z \\ \rightarrow \neg(\textstyle\bigwedge_{i=1}^{n+1} \bigwedge_{j=1, j\neq i}^{n+1} \neg(z_i = z_j))))).\end{aligned}$$

Even in this case, condition (1) on quantifiers is satisfied as well. This can be checked for both the conjuncts

$(\forall z_1)\dots(\forall z_n)(\bigwedge_{i=1}^{n} z_i \in Z \to \bigwedge_{i=1}^{n} z_i \in X_i)$, and
$(\forall z_1)\dots(\forall z_{n+1})(\bigwedge_{i=1}^{n+1} z_i \in Z \to \neg(\bigwedge_{i=1}^{n+1} \bigwedge_{j=1,j\neq i}^{n+1} \neg(z_i = z_j)))$

just as in the previous examples.

Another interesting variant of the power set is the $\mathrm{pow}^*(X_1,\dots,X_n)$, which denotes the collection

$$\{Z : Z \subseteq \bigcup_{i=1}^{n} X_i \text{ and } Z \cap X_i \neq \emptyset, \text{ for all } 1 \leq i \leq n\}$$

introduced in [4]. The literal $A = \mathrm{pow}^*(X_1,\dots,X_n)$ is expressed in $3LQS^R$ by

$$(\forall Z)(Z \in A \leftrightarrow ((\forall z)(z \in Z \to z \in \bigvee_{i=1}^{n} z \in X_i) \wedge \bigwedge_{i=1}^{n} \neg(\forall z)(z \in Z \to \neg z \in X_i)).$$

Also with this formula one can verify that the restriction on quantifiers is satisfied by checking the subformulae:

$(\forall z)(z \in Z \to z \in \bigvee_{i=1}^{n} z \in X_i)$,
$(\forall z)(z \in Z \to \neg z \in X_i)$, for $i = 1,\dots,n$.

5.4 Other applications of $3LQS^R$

In this section we introduce a family $\{(3LQS^R)^h\}_{h\geq 2}$ of sublanguages of $3LQS^R$, each having the satisfiability problem NP-complete. Then, in Section 5.4 we illustrate how the modal logic S5 can be expressed by the language $(3LQS^R)^3$.

Formulae in $(3LQS^R)^h$ must satisfy several syntactic constraints, as specified in Definition 3 below, that are crucial to establish NP-completeness of the satisfiability problem for the language, specifically to show that it is in NP. First of all, the length of all quantifier prefixes occurring in a formula of $(3LQS^R)^h$ must be bounded by the constant h. Thus, given a satisfiable $(3LQS^R)^h$-formula φ and a $3LQS^R$-model $\boldsymbol{\mathcal{M}} = (D, M)$ for it, from Theorem 1 it follows that φ is satisfied by the relativized interpretation $\boldsymbol{\mathcal{M}}^* = (D^*, M^*)$ of $\boldsymbol{\mathcal{M}}$ with respect to a domain D^* having its size bounded as specified in (2). Since $\max m$ and $\max n$ occurring in (2) are bounded by the constant h, it follows that the bound expressed in (2) is polynomial in the size of φ. The other syntactic constraints on $(3LQS^R)^h$-formulae are introduced to deduce that $M^*A \subseteq \mathrm{pow}_{\pi,h}(D^*)$, for any free variable A of sort 2 in φ, so that the model $\boldsymbol{\mathcal{M}}^*$ can be guessed in nondeterministic polynomial time in the size of φ, and the fact that $\boldsymbol{\mathcal{M}}^*$ actually satisfies φ can be verified in deterministic polynomial time. This is enough to prove that the satisfiability problem for $(3LQS^R)^h$-formulae is in NP.

Definition 3 (**$(3LQS^R)^h$-formulae**). *Let φ be a $3LQS^R$-formula and let $A_1,\dots,A_p$ be all the variables of sort 2 occurring in it. Then φ is a $(3LQS^R)^h$-formula, with $h \geq 2$, if it has the form*

$$\xi_U \wedge \xi_{\pi,h} \wedge \psi_1 \wedge \dots \wedge \psi_p \wedge \chi\,,$$

where

1. $\xi_U \equiv (\forall z)(z \in X_U)$
i.e., X_U is the (nonempty) universe of discourse*;*

2. $\xi_{\pi,h} \equiv (\forall Z)\left(Z \in A_{\pi,h} \leftrightarrow (\forall z_1)\dots(\forall z_h)\left(\bigwedge_{i=1}^{h} z_i \in Z \rightarrow \bigvee_{i,j=1,i<j}^{h} z_i = z_j\right)\right)$
i.e., $A_{\pi,h} = \mathrm{pow}_{<h}(X_U)$ (together with formula ξ_U)*;*

3. $\psi_i \equiv (\forall Z)(Z \in A_i \rightarrow Z \in A_{\pi,h})$*, for* $i = 1, \dots, p$*;*
i.e., $A_i \subseteq \mathrm{pow}_{\pi h}(X_U)$, for $i = 1, \dots, p$ (together with formulae ξ_U and $\xi_{\pi,h}$)*;*

4. χ *is a propositional combination of*
(a) quantifier-free atomic formulae of any level,
(b) quantified atomic formulae of level 1 of the form

$$(\forall z_1)\dots(\forall z_n)\varphi_0\,,$$

with $n \leq h$*,*

(c) quantified atomic formulae of level 2 of the form

$$(\forall Z_1)\dots(\forall Z_m)((Z_1 \in A_{\pi,h} \wedge \dots \wedge Z_m \in A_{\pi,h}) \rightarrow \varphi_1)\,,$$

where $m \leq h$ *and* φ_1 *is a propositional combination of quantifier-free atomic formulae and of quantified atomic formulae of level 1 satisfying (4b) above.*

Next we give the following complexity result on $(3LQS^R)^h$.

Theorem 2. *The satisfiability problem for* $(3LQS^R)^h$ *is* NP*-complete.*

Proof. NP-hardness of our problem can be proved by reducing an instance of the satisfiability problem for propositional logic to our problem.

We prove that our problem is in NP reasoning as follows. Let

$$\varphi \equiv \xi_U \wedge \xi_{\pi,h} \wedge \psi_1 \wedge \dots \wedge \psi_p \wedge \chi \tag{5}$$

be a satisfiable $(3LQS^R)^h$-formula, and let H_φ be a set of formulae defined as follows. Initially, we put

$$H_\varphi := \{\xi_U, \xi_{\pi,h}, \psi_1, \dots, \psi_p, \chi\}$$

and then, we modify H_φ according to the following five rules, until no rule can be further applied:

R1: if $\xi \equiv \neg\neg\xi_1$ is in H_φ, then $H_\varphi = (H_\varphi \setminus \{\xi\}) \cup \{\xi_1\}$,

R2: if $\xi \equiv \xi_1 \wedge \xi_2$ (resp., $\xi \equiv \neg(\xi_1 \vee \xi_2)$) is in H_φ (i.e., ξ is a conjunctive formula), then we put $H_\varphi := (H_\varphi \setminus \{\xi\}) \cup \{\xi_1, \xi_2\}$ (resp., $H_\varphi := (H_\varphi \setminus \{\xi\}) \cup \{\neg\xi_1, \neg\xi_2\}$),

R3: if $\xi \equiv \xi_1 \vee \xi_2$ (resp., $\xi \equiv \neg(\xi_1 \wedge \xi_2)$) is in H_φ (i.e., ξ is a disjunctive formula), then we choose a ξ_i, $i \in \{1, 2\}$, such that $H_\varphi \cup \{\xi_i\}$ (resp., $H_\varphi \cup \{\neg\xi_i\}$) is satisfiable and put $H_\varphi := (H_\varphi \setminus \{\xi\}) \cup \{\xi_i\}$ (resp., $H_\varphi := (H_\varphi \setminus \{\xi\}) \cup \{\neg\xi_i\}$),

R4: if $\xi \equiv \neg(\forall z_1)\dots(\forall z_n)\varphi_0$ is in H_φ, then $H_\varphi := (H_\varphi \setminus \{\xi\}) \cup \{\neg(\varphi_0)^{z_1,\dots,z_n}_{\bar{z}_1,\dots,\bar{z}_n}\}$, where $\bar{z}_1, \dots, \bar{z}_n$ are newly introduced variables of sort 0,

R5: if $\xi \equiv \neg(\forall Z_1)\dots(\forall Z_m)\varphi_1$ is in H_φ, then $H_\varphi := (H_\varphi \setminus \{\xi\}) \cup \{\neg(\varphi_1)^{Z_1,\dots,Z_m}_{\bar{Z}_1,\dots,\bar{Z}_m}\}$, where $\bar{Z}_1,\dots,\bar{Z}_m$ are new variables of sort 1.

It is easy to see that the above construction terminates in $\mathcal{O}(|\varphi|)$ steps. Let us put $\psi \equiv \bigwedge_{\xi \in H_\varphi} \xi$. Clearly

(a) ψ is a satisfiable $(3LQS^R)^h$-formula,
(b) $|\psi| = \mathcal{O}(|\varphi|)$, and
(c) $\psi \to \varphi$ is a valid $3LQS^R$-formula.

In the light of (a)–(c) above, to prove that our problem is in NP, we only have to construct in nondeterministic polynomial time a $3LQS^R$-interpretation and show that we can check in polynomial time that it actually satisfies ψ.

Let $\boldsymbol{\mathcal{M}} = (D, M)$ be a $3LQS^R$-model for ψ and let $\boldsymbol{\mathcal{M}}^* = (D^*, M^*)$ be the relativized interpretation of $\boldsymbol{\mathcal{M}}$ with respect to a domain D^*, hence such that $|D^*| = \mathcal{O}(|\psi|^{h+1})$, since ψ is a $(3LQS^R)^h$-formula (cf. Theorem 1 and the construction described in Sections 4.2 and 3).

In the light of the remarks just before Definition 3, to complete our proof we just have to verify that

- $M^*A \subseteq \mathrm{pow}_{<h}(D^*)$, for any free variable A of sort 2 in ψ (which entails that $|M^*A| = \mathcal{O}(|D^*|^h)$), and
- $\boldsymbol{\mathcal{M}}^* \models \psi$ can be verified in deterministic polynomial time.

The first statement can easily be checked making the following considerations. By the formula $\xi_{\pi,h}$, we have that $M^*A_{\pi,h} = \mathrm{pow}_{<h}(D^*)$. Concerning the other A_is of sort 2 occurring in ψ, we just have to notice that ψ must contain a conjunct ψ_i associated to A_i that, together with $\xi_{\pi,h}$ ensures that $M^*A_i \subseteq \mathrm{pow}_{<h}(D^*)$. The proof of the second statement follows from the fact that each quantified subformula of ψ has the quantifier prefix bounded by h and that each quantified formula of level 2 has its quantified variables of level 1 ranging in $\mathrm{pow}_{<h}(D^*)$.

Hence the satisfiability problem for $(3LQS^R)^h$-formulae is in NP, and since it is also NP-hard, it follows that it is NP-complete.

In the next section we will show how the $3LQS^R$ fragment can be used to formalize the modal logic $\mathsf{S5}$.

Formalization of S5 in $3LQS^R$ Let us start with some preliminary notions on modal logics. The *modal language* L_M is based on a countably infinite set of propositional letters $\mathcal{P} = \{p_1, p_2, \dots\}$, the classical propositional connectives '$\neg$', '$\wedge$' , and '$\vee$', the modal operators '$\Box$', '$\Diamond$' (and the parentheses). L_M is the smallest set such that $\mathcal{P} \subseteq \mathsf{L}_M$, and such that if $\varphi, \psi \in \mathsf{L}_M$, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\Box\varphi$, $\Diamond\varphi \in \mathsf{L}_M$. Lower case letters like p denote elements of $\mathcal{P}$ and Greek letters like φ and ψ represent formulae of L_M. Given a formula φ of L_M, we indicate with $SubF(\varphi)$ the set of the subformulae of φ.

A *normal modal logic* is any subset of L_M which contains all the tautologies and the axiom

$$\mathsf{K}: \ \Box(p_1 \to p_2) \to (\Box p_1 \to \Box p_2)\,,$$

Axiom	Schema	Condition on R
T	$\Box p \rightarrow p$	Reflexive
5	$\Diamond p \rightarrow \Box\Diamond p$	Euclidean
B	$p \rightarrow \Box\Diamond p$	Symmetric
4	$\Box p \rightarrow \Box\Box p$	Transitive
D	$\Box p \rightarrow \Diamond p$	Serial: $(\forall w)(\exists u)R(w,u)$

Table 4. Axioms of normal modal logics

and which is closed with respect to modus ponens, substitution, and necessitation (the reader may consult a text on modal logic like [3] for more details).

A *Kripke frame* is a pair $\langle W, R\rangle$ such that W is a nonempty set of possible worlds and R is a binary relation on W called *accessibility relation*. If $R(w,u)$ holds, we say that the world u is accessible from the world w. A *Kripke model* is a triple $\langle W, R, h\rangle$, where $\langle W, R\rangle$ is a Kripke frame and h is a function mapping propositional letters into subsets of W. Thus, $h(p)$ is the set of all the worlds where p is true.

Let $K = \langle W, R, h\rangle$ be a Kripke model and let w be a world in K. Then, for every $p \in \mathcal{P}$ and for every $\varphi, \psi \in \mathsf{L}_M$, the relation of satisfaction $\models$ is defined as follows:

- $K, w \models p$ iff $w \in h(p)$;
- $K, w \models \varphi \vee \psi$ iff $K, w \models \varphi$ or $K, w \models \psi$;
- $K, w \models \varphi \wedge \psi$ iff $K, w \models \varphi$ and $K, w \models \psi$;
- $K, w \models \neg\varphi$ iff $K, w \not\models \varphi$;
- $K, w \models \Box\varphi$ iff $K, w' \models \varphi$, for every $w' \in W$ such that $(w, w') \in R$;
- $K, w \models \Diamond\varphi$ iff there is a $w' \in W$ such that $(w, w') \in R$ and $K, w' \models \varphi$.

A formula φ is said to be *satisfied* at w in K if $K, w \models \varphi$; φ is said to be *valid* in K (and we write $K \models \varphi$), if $K, w \models \varphi$, for every $w \in W$.

The smallest normal modal logic is K, which contains only the modal axiom K and whose accessibility relation R can be any binary relation. The other normal modal logics admit together with K other modal axioms drawn from the ones in Table 4.

In this paper we analyze the modal logic $\mathsf{S5}$ which is the strongest normal modal system. It can be obtained from the logic K in several ways. One of them consists in adding axioms T and **5** from Table 4 to the logic K. Given a formula φ, a Kripke model $K = \langle W, R, h\rangle$, and a world $w \in W$, the semantics of the modal operators can be defined as follows:

- $K, w \models \Box\varphi$ iff $K, v \models \varphi$, for every $v \in W$,
- $K, w \models \Diamond\varphi$ iff $K, v \models \varphi$, for some $v \in W$.

This makes it possible to translate a formula φ of $\mathsf{S5}$ into the $3LQS^R$ language.

For the purpose of simplifying the definition of the translation function τ_{S5} given below, the concept of "empty formula" is introduced, to be denoted by Λ, and not interpreted in any particular way. The only requirement on Λ needed

for the definition given next is that $\Lambda \wedge \psi$ and $\psi \wedge \Lambda$ are to be considered as syntactic variations of ψ, for any $3LQS^R$-formula ψ.

For every propositional letter p, let $\tau^1_{\mathsf{S5}}(p) = X^1_p$, where $X^1_p \in \mathcal{V}_1$, and let $\tau^2_{\mathsf{S5}} : \mathsf{S5} \to 3LQS^R$ be the function defined recursively as follows:

- $\tau^2_{\mathsf{S5}}(p) = \Lambda$,
- $\tau^2_{\mathsf{S5}}(\neg\varphi) = (\forall z)(z \in X^1_{\neg\varphi} \leftrightarrow \neg(z \in X^1_{\varphi})) \wedge \tau^2_{\mathsf{S5}}(\varphi)$,
- $\tau^2_{\mathsf{S5}}(\varphi_1 \wedge \varphi_2) = (\forall z)(z \in X^1_{\varphi_1 \wedge \varphi_2} \leftrightarrow (z \in X^1_{\varphi_1} \wedge z \in X^1_{\varphi_2})) \wedge \tau^2_{\mathsf{S5}}(\varphi_1) \wedge \tau^2_{\mathsf{S5}}(\varphi_2)$,
- $\tau^2_{\mathsf{S5}}(\varphi_1 \vee \varphi_2) = (\forall z)(z \in X^1_{\varphi_1 \vee \varphi_2} \leftrightarrow (z \in X^1_{\varphi_1} \vee z \in X^1_{\varphi_2})) \wedge \tau^2_{\mathsf{S5}}(\varphi_1) \wedge \tau^2_{\mathsf{S5}}(\varphi_2)$,
- $\tau^2_{\mathsf{S5}}(\Box\varphi) =$ $(\forall z)(z \in X^1_{\varphi}) \to (\forall z)(z \in X^1_{\Box\varphi}) \wedge \neg(\forall z)(z \in X^1_{\varphi}) \to (\forall z)\neg(z \in X^1_{\Box\varphi}) \wedge \tau^2_{\mathsf{S5}}(\varphi)$,
- $\tau^2_{\mathsf{S5}}(\Diamond\varphi) =$ $\neg(\forall z)\neg(z \in X^1_{\varphi}) \to (\forall z)(z \in X^1_{\Diamond\varphi}) \wedge (\forall z)\neg(z \in X^1_{\varphi}) \to (\forall z)\neg(z \in X^1_{\Diamond\varphi}) \wedge \tau^2_{\mathsf{S5}}(\varphi)$,

where Λ is the empty formula and $X^1_{\neg\varphi}, X^1_{\varphi}, X^1_{\varphi_1 \wedge \varphi_2}, X^1_{\varphi_1 \vee \varphi_2}, X^1_{\varphi_1}, X^1_{\varphi_2} \in \mathcal{V}_1$.

Finally, for every φ in $\mathsf{S5}$, if φ is a propositional letter in $\mathcal{P}$ we put $\tau_{\mathsf{S5}}(\varphi) = \tau^1_{\mathsf{S5}}(\varphi)$, otherwise $\tau_{\mathsf{S5}}(\varphi) = \tau^2_{\mathsf{S5}}(\varphi)$.

Even though the accessibility relation R is not used in the translation, we can give its formalization in the $3LQS^R$ fragment by introducing the collection variable A_R and the following related formulae:

- $\psi_1 = (\forall Z)(Z \in A_R \to Z \in A_{\pi,3})$,
- $\chi_1 = (\forall Z)(Z \in A_{\pi,3} \to (Z \in A_R \leftrightarrow (\forall z_1)(\forall z_2)(\forall z_3)((z_1 \in Z \wedge z_2 \in Z \wedge z_3 \in Z) \to (z_1 = z_2 \vee z_2 = z_3 \vee z_1 = z_3))$.

Clearly $\tau_{\mathsf{S5}}(\varphi)$ and the formulae above belong to $3LQS^R$ and, in particular, to $(3LQS^R)^3$. Correctness of the above translation is guaranteed by the following lemma.

Lemma 5. *For every formula φ of the logic* $\mathsf{S5}$*, φ is satisfiable in a model $K = \langle W, R, h\rangle$ iff there is a $3LQS^R$-interpretation satisfying $x \in X_\varphi$.* □

Proof. Let $\bar{w}$ be a world in W. We construct a $3LQS^R$-interpretation $\boldsymbol{\mathcal{M}} = (W, M)$ as follows:

- $Mx = \bar{w}$,
- $MX^1_p = h(p)$, where p is a propositional letter and $X^1_p = \tau_{\mathsf{S5}}(p)$,
- $M\tau_{\mathsf{S5}}(\psi) = \mathbf{true}$, for every $\psi \in SubF(\varphi)$, where ψ is not a propositional letter.

To prove the lemma, it would be enough to show that $K, \bar{w} \models \varphi$ iff $M \models x \in X^1_\varphi$. However, it is more convenient to prove the following more general property:

Given a $w \in W$, if $y \in \mathcal{V}_0$ is such that $My = w$, then

$$K, w \models \varphi \textit{ iff } M \models y \in X^1_\varphi,$$

which we do by structural induction on φ.

Base case: If φ is a propositional letter, by definition, $K, w \models \varphi$ iff $w \in h(\varphi)$. But this holds iff $My \in MX^1_\varphi$, which is equivalent to $M \models y \in X^1_\varphi$.

Inductive step: We consider only the cases in which $\varphi = \Box\psi$ and $\varphi = \Diamond\psi$, as the other cases can be dealt with similarly.

- If $\varphi = \Box\psi$, assume first that $K, w \models \Box\psi$. Then $K, w \models \psi$ and, by inductive hypothesis, $M \models y \in X^1_\psi$. Since $M \models \tau_{\mathsf{S5}}(\Box\psi)$, it holds that $M \models (\forall z_1)(z_1 \in X^1_\psi) \rightarrow (\forall z_2)(z_2 \in X^1_{\Box\psi})$. Then we have $M[z_1/w, z_2/w] \models (z_1 \in X^1_\psi) \rightarrow (z_2 \in X^1_{\Box\psi})$ and, since $My = w$, we have also that $M \models (y \in X^1_\psi) \rightarrow (y \in X^1_{\Box\psi})$. By the inductive hypothesis and by modus ponens we obtain $M \models y \in X^1_{\Box\psi}$, as required.
  On the other hand, if $K, w \not\models \Box\psi$, then $K, w \not\models \psi$ and, by inductive hypothesis, $M \not\models y \in X^1_\psi$. Since $M \models \tau_{\mathsf{S5}}(\Box\psi)$, then $M \models \neg(\forall z_1)(z_1 \in X^1_\psi) \rightarrow (\forall z_2)\neg(z_2 \in X^1_{\Box\psi})$. By the inductive hypothesis and some predicate logic manipulations, we have $M \models \neg(y \in X^1_\psi) \rightarrow \neg(y \in X^1_{\Box\psi})$, and by modus ponens we infer $M \models \neg(y \in X^1_{\Box\psi})$, as we wished to prove.
- Let $\varphi = \Diamond\psi$ and, to begin with, assume that $K, w \models \Diamond\psi$. Then, there is a w' such that $K, w' \models \psi$, and a $y' \in \mathcal{V}_0$ such that $My' = w'$. Thus, by inductive hypothesis, we have $M \models y' \in X^1_\psi$ and, by predicate logic, $M \models \neg(\forall z_1)\neg(z_1 \in X^1_\psi)$. By the very definition of M, $M \models \tau_{\mathsf{S5}}(\Diamond\psi)$ and thus $M \models \neg(\forall z_1)\neg(z_1 \in X^1_\psi) \rightarrow (\forall z_2)(z_2 \in X^1_{\Diamond\psi})$. Then, by modus ponens we obtain $M \models (\forall z_2)(z_2 \in X^1_{\Diamond\psi})$ and finally, by predicate logic, $M \models y \in X^1_{\Diamond\psi}$.
  On the other hand, if $K, w \not\models \Diamond\psi$, then $K, w' \not\models \psi$, for any $w' \in W$ and, since $w' = My'$ for any $y' \in \mathcal{V}_0$, it holds that $M \not\models y' \in X^1_\psi$ and thus, by predicate logic, $M \models (\forall z_1)\neg(z_1 \in X^1_\psi)$.
  Reasoning as above, $M \models (\forall z_1)\neg(z_1 \in X^1_\psi) \rightarrow (\forall z_2)\neg(z_2 \in X^1_{\Diamond\psi})$ and, by modus ponens, $M \models (\forall z_2)\neg(z_2 \in X^1_{\Diamond\psi})$. Finally, by predicate logic, $M \not\models y \in X^1_{\Diamond\psi}$, as required.

It can be checked that $\tau_{\mathsf{S5}}(\varphi)$ is polynomial in the size of φ and that its satisfiability can be verified in nondeterministic polynomial time since the formula $\xi_W \wedge \xi_{\pi,3} \wedge \psi_1 \wedge (\chi_1 \wedge \tau_{\mathsf{S5}}(\varphi))$, where ξ_W denotes W, and the other conjuncts are as defined above, belongs to $(3LQS^R)^3$. Consequently, considering that $\mathsf{S5}$ was proved NP-complete in [14], the decision algorithm presented in this paper together with the translation function introduced above can be considered an optimal procedure (in terms of its computational complexity class) to decide the satisfiability of any formula φ of $\mathsf{S5}$.

6 Conclusions and future work

We have presented the three-sorted stratified set-theoretic fragment $3LQS^R$ and have given a decision procedure for its satisfiability problem. Then, we singled out a family of sublanguages of $3LQS^R$, $\{(3LQS^R)^h\}_{h\geq 2}$, characterized by imposing further constraints in the construction of the formulae, we proved that each language in the family has the satisfiability problem NP-complete, and we showed that the modal logic $\mathsf{S5}$ can be formalized in $(3LQS^R)^3$.

Techniques to translate modal formulae in set theoretic terms have already been proposed in [1], in the context of hyperset theory, and in [11] in the ambit of weak set theories not involving the axiom of extensionality and the axiom of foundation.

We further intend to study the possibility of formulating non-classical logics in the context of well-founded set theory constructing suitable extensions of the $3LQS^R$ fragment. In particular, we plan to introduce in our language a notion of ordered pair and the operation of composition for binary relations.

We also plan to extend the language so as it can express the set theoretical construct of general union, thus being able to subsume the theory $3LSSPU$. Another direction of future investigations concerns n-sorted languages involving also constructs to express ordered n-uples of individuals.

References

1. J. Barwise and L. Moss. *Vicious circles.* Vol. 60 of *CSLI Lecture notes.* CSLI, Stanford, CA, 1996.
2. N. S. Bjørner, A. Browne, E. S. Chang, M. Colón, A. Kapur, Z. Manna, H. B. Sipma, and T. E. Uribe. STeP: Deductive-algorithmic verification of reactive and real-time systems. In *Proc. Intl. Conference on Computer Aided Verification*, volume 1102 of LNCS, pages 415–418. Springer, 1996.
3. P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic.* Cambridge University Press, Cambridge Tracts in Theoretical Computer Science, 2001.
4. D. Cantone. Decision procedures for elementary sublanguages of set theory: X. Multilevel syllogistic extended by the singleton and powerset operators. *Journal of Automated Reasoning*, volume 7, number 2, pages 193–230, Kluwer Academic Publishers, Hingham, MA, USA, 1991.
5. D. Cantone and V. Cutello. A decidable fragment of the elementary theory of relations and some applications. In *ISSAC '90: Proceedings of the international symposium on Symbolic and algebraic computation*, pages 24–29, New York, NY, USA, 1990. ACM Press.
6. D. Cantone and V. Cutello. Decision procedures for stratified set-theoretic syllogistics. In Manuel Bronstein, editor, *Proceedings of the 1993 International Symposium on Symbolic and Algebraic Computation, ISSAC'93 (Kiev, Ukraine, July 6-8, 1993)*, pages 105–110, New York, 1993. ACM Press.
7. D. Cantone, V. Cutello, and J. T. Schwartz. Decision problems for Tarski and Presburger arithmetics extended with sets. In *CSL '90: Proceedings of the 4th Workshop on Computer Science Logic*, pages 95–109, London, UK, 1991. Springer-Verlag.

8. D. Cantone and A. Ferro Techniques of computable set theory with applications to proof verification. *Comm. Pure Appl. Math.*, pages 901–945, vol. XLVIII, 1995. Wiley.
9. D. Cantone, A. Ferro, and E. Omodeo. *Computable set theory.* Clarendon Press, New York, NY, USA, 1989.
10. D. Cantone, E. Omodeo, and A.Policriti. *Set Theory for Computing - From decision procedures to declarative programming with sets.* Springer-Verlag, Texts and Monographs in Computer Science, 2001.
11. G. D'Agostino, A. Montanari, and A. Policriti. A set-theoretic translation method for polimodal logics. *Journal of Automated Reasoning*, 3(15): 317–337, 1995.
12. A. Ferro and E.G. Omodeo. *An efficient validity test for formulae in extensional two-level syllogistic.* Le Matematiche, 33:130–137, 1978.
13. A. Ferro, E.G. Omodeo and J.T. Schwartz. *Decision procedures for elementary sublanguages of set theory. I. Multi-level syllogistic and some extensions.* Communication on Pure and Applied Mathematics, 33, 599–608, 1980.
14. R. Ladner. The computational complexity of provability in systems of modal propositional logic. *SIAM Journal of Computing,* 6: 467-480, 1977.
15. E. G. Omodeo, D. Cantone, A. Policriti, and J. T. Schwartz. A Computerized Referee. In *Reasoning, Action and Interaction in AI Theories and Systems*, Lecture Notes in Artificial Intelligence, vol. 4155, pages 117–139, 2006.
16. J. Schwartz, D. Cantone, and E. Omodeo. *Computational Logic and Set Theory – Applying Formalized Logic to Analysis.* Springer 2011, I-XVII, 1-416.